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Integrable non-linear evolutions in 2+1 dimensions with non-analytic dispersion relations

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Abstract. We develop a method of obtaining two-dimensional integrable evolutions corresponding to singular dispersion relations. The method is applied to the 2×2 linear first-order eigenvalue problem and the inverse spectral transform scheme is established. The Bäcklund transformation and non-linear superposition formula are used to obtain the soliton solutions.

1. Introduction

We are interested here in building non-linear evolutions in $2+1$ dimensions $(x, y; t)$ corresponding to singular dispersion relations (non-analytic functions of the spectral parameter).

The differential analysis in the complex plane ($\bar{\partial}$ method) which has been applied for solving non-linear evolutions in $2+1$ dimensions with polynomial dispersion relations [1-6] has proved to be very useful in considering, in $1+1$ dimensions, the classes of evolutions having a non-analytic dispersion relation [7].

The spectral transform method has been known since 1973 to be applicable to evolutions with singular dispersion relations [8-10], and to lead to physically relevant equations (Maxwell-Bloch equations for a medium consisting of inhomogeneously broadened and non-degenerate two-level atoms). Recently, however, a new solvable evolution of this class has been found [11]. Using the method of [7] it is now possible to build such evolutions very easily, together with their Bäcklund transformations [7, 12].

We show here that the approach can be extended to $2+1$ dimensions where the evolutions still present the feature of a system of coupled differential equations with, however, a more complicated dependence on the spectral parameter involving Cauchy-like integrals.

The main tool is the $\bar{\partial}$ approach of solvable evolutions as systematically developed in [13, 14] which makes explicit use of the asymptotic expansions in the neighbourhood of the boundaries of the complex domain.

The general method is presented in § 2 where the construction of a Lax pair (T_1, T_2) of commuting spectral operators is performed. The principal spectral operator T_1 is chosen to be the generalisation of the Zakharov-Shabat operator in the plane [4]. The

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auxiliary spectral operator T_2, which fixes the time evolution, is obtained by means of the solution of a linear singular integral equation. This has to be contrasted with the situation in $1+1$ dimensions where T_2 is explicitly given [7].

In § 3 we obtain explicit solvable non-linear evolutions which in one spatial dimension reduce to the 'self-induced transparency' [8] or 'reduced Maxwell-Bloch' [9] equations. These new two-dimensional solvable evolutions arise then as natural extensions of physically relevant systems and do show the same essential property of having exponential damping (attenuator [10]) or growth (amplifier [15, 16]) of the radiative part of the spectral transform.

In §§ 4 and 5 we examine in more detail the evolution equations and the evolution of the spectral transform when the chosen $\bar{\partial}$ problem leads to either a hyperbolic differential operator T_1 or an elliptic one.

The soliton-like solutions are obtained in § 6 which is devoted to the study of the Bäcklund transformations (BT). While the spatial part of the BT, being obtained from T_1, is obviously the same as for polynomial dispersion relations, the method for deriving and solving the time part is quite different. It requires us to obtain first from the spatial part not only the potential but also the corresponding spectral transform. It is then by making this spectral transform evolve in time that the time part of the BT can be solved. We derive in this way the elementary BT which allows us to build a whole lattice of solutions by using the non-linear superposition principle [17].

2. General method

In the spectral transform method, the usually adopted approach consists first in giving a differential (with respect to the xy space variables) linear operator T_1 together with its spectral eigenspace of function $\psi(k, x, y)$, for instance (matrix case)

$$T_1\psi = 0 \qquad \psi(k, x, y)E^{-1}(k, x, y) \to \mathbb{1} \qquad \text{as } x \to -\infty \tag{2.1}$$

where E is some given function which solves

$$\{\lim_{x\to-\infty} T_1\}E = 0. \tag{2.2}$$

Note that here E allows us to introduce the k dependence of the eigenfunctions ψ and therefore to set a spectral problem for a differential operator. Second, the standard approach consists in seeking operators T_2 (involving the additional variable t) for which is required either the compatibility condition

$$[T_1, T_2] = 0 \tag{2.3}$$

or its weak version [5, 6]

$$[T_1, T_2]\psi = 0. \tag{2.4}$$

From the works quoted in [1-6] we know that the spectral problem (2.1) related to T_1 can be written and solved as a $\bar{\partial}$ problem on a complex domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$:

$$\frac{\partial}{\partial\bar{k}}\psi(k, x, y) = \int\int_{\mathcal{D}} \mathrm{d}l \wedge \mathrm{d}\bar{l}\,\psi(l, x, y)R(k, l) \qquad k \in \mathcal{D} \tag{2.5a}$$

$$\psi(k, x, y)E^{-1}(k, x, y) \sim \mathbb{1} \qquad k \to \partial\mathcal{D}. \tag{2.5b}$$

The matrix $R(k, l)$ is a distribution in $\mathscr{D} \times \mathscr{D}$ and E analytic in $\mathscr{D}$. We note that here the function $E(k, x, y)$ is the tool for introducing an xy dependence in the solution ψ of (2.5) which, according to the Cauchy-Green formula, is

$$\psi(k, x, y) = E(k, x, y) + \frac{1}{2i\pi} \int\int_{\mathscr{D}} \frac{d\xi \wedge d\bar{\xi}}{\xi - k} \frac{\partial \psi}{\partial \bar{\xi}} E^{-1}(\xi, x, y) E(k, x, y). \tag{2.6}$$

Instead of the spectral problem (2.1), we adopt here as a starting point the $\bar{\partial}$ problem (2.5) and introduce a time dependence by prescribing the following linear evolution:

$$R_t(k, l; t) = R(k, l; t)\Omega(k; t) - \Omega(l; t)R(k, l; t). \tag{2.7}$$

In the above equation $\Omega(k; t)$ is a given matrix-valued function on $\mathscr{D} \times \mathbb{R}$ and is called the dispersion relation. It is worth mentioning that the t dependence is parametric in the sense that the basic equations (2.5) are maintained valid for any time t. In particular it is essential to take E to be time independent.

Now, given the $\bar{\partial}$ problem (2.5) and the evolution (2.7), in other words given $R(k, l; 0)$, $E(k, x, y)$ and $\Omega(k, t)$, we want to build a couple of compatible operators T_1 and T_2 (obeying (2.4)). To that end, an essential assumption is that the $\bar{\partial}$ problem (2.5), or preferably the integral equation (2.6), has a unique solution. Only for some known examples is one able to give sufficient conditions on $R(k, l)$ to realise this requirement. But the general problem (or characterisation problem) of giving necessary and sufficient conditions such that (2.6) has a unique solution is not solved.

Then, given the $\bar{\partial}$ problem (2.5) with some explicit function $E(k, x, y)$, T_1 can be constructed by following the method of [13]. This will be done for completeness in the next section.

We seek now T_2 under the form

$$T_2 = \partial_t - W \tag{2.8}$$

where $W = W(k, x, y, t)$ is a matrix to be determined, with the requirement

$$T_2\psi = \psi\Omega. \tag{2.9}$$

The above condition ensures in particular that the weak compatibility condition (2.4) is satisfied.

By differentiating (2.9) with respect to $\bar{k}$ and by using (2.5) and (2.7) we obtain for $k \in \mathscr{D}$:

$$\frac{\partial W}{\partial \bar{k}} = -\psi(k) \frac{\partial \Omega}{\partial \bar{k}} \psi^{-1}(k) + \int\int_{\mathscr{D}} dl \wedge d\bar{l}[W(l) - W(k)]\psi(l)R(k, l)\psi^{-1}(k). \tag{2.10}$$

In the usual cases Ω is a polynomial in k ($\partial\Omega/\partial\bar{k} = 0$) and W is a differential operator; therefore (2.10) would be irrelevant and to compute T_2 one should instead solve directly (2.3) or (2.4). Also (2.9) would *imply* the evolution (2.7) because the operators T_2 and $\partial\bar{k}$ would commute (apply T_2 on both sides of (2.5a)).

Here, however, we are interested in the cases when the dispersion relation Ω is singular, i.e. when there exists a subset $\mathscr{E}$ of $\mathscr{D}$ for which

$$\frac{\partial \Omega}{\partial \bar{k}} \neq 0 \qquad k \in \mathscr{E}. \tag{2.11}$$

Then (2.10) allows us to reconstruct W provided its behaviour on the boundary $\partial\mathscr{D}$ is given. This implies some conditions on Ω which depend on the chosen example and will be given later.

In summary, for non-analytic dispersion relations Ω, we are able to build from the $\bar\partial$ problem a pair of operators T_1 and T_2. Then the weak commutativity condition (2.4) which is

$$\{T_{1,t}+[T_1, W]\}\psi = 0 \tag{2.12}$$

is identically satisfied in $\mathscr{D}$. In particular it furnishes the k-independent equation

$$\lim_{k\to\partial\mathscr{D}} \{(T_{1,t}\psi+[T_1, W]\psi)\psi^{-1}\} = 0. \tag{2.13}$$

It is the solvable non-linear evolution which we were seeking and it is coupled to the spectral problem (2.1) through the solution W of (2.10). We shall see for an example in the next section that it is actually more convenient to replace the spectral problem with the following equation:

$$\frac{\partial}{\partial\bar k}\{(T_{1,t}\psi+[T_1, W]\psi)\psi^{-1}\} = 0. \tag{2.14}$$

We remark finally that the one-dimensional case corresponds to a local $\bar\partial$ problem

$$R(k, l) = R_0(k)\delta(l-k) \tag{2.15}$$

and hence (2.10) becomes simply

$$\frac{\partial W}{\partial\bar k} = -\psi\frac{\partial\Omega}{\partial\bar k}\psi^{-1} \tag{2.16}$$

which allows us to build the operator T_2 in closed form [7].

3. An example of non-linear solvable evolution

To give an example it is necessary to choose in the $\bar\partial$ problem (2.5) both the domain $\mathscr{D}$ and the function E. Let

$$\mathscr{D} = \mathbb{C} \qquad \partial\mathscr{D} = \{k/|k| = \infty\} \tag{3.1}$$

$$E(k, x, y) = \exp[ik(\sigma_3 x - \varepsilon y)] \qquad \varepsilon^2 = \pm 1. \tag{3.2}$$

We recall now the method [13] for obtaining the operator T_1 from the $\bar\partial$ problem (2.5) with the above boundary conditions.

The integral equation (2.6) provides the expansion

$$\psi(k, x, y) \sim \left(\mathbb{1} + \sum_{n=1}^{\infty} k^{-n}\phi^{(n)}(x, y)\right)\exp[ik(\sigma_3 x - \varepsilon y)]. \tag{3.3}$$

By assumption, $R(k, l)$ in (2.5) does not depend on the space variables and therefore ψ_x and ψ_y are also solutions of the $\bar\partial$ equation (2.5a) with, however, a different behaviour as $|k|\to\infty$. The problem consists in finding a linear combination of ψ_x and ψ_y whose asymptotic behaviour can be related to that of ψ, independent of k. We obtain

$$\psi_x \pm \varepsilon\sigma_3\psi_y \sim i[\phi^{(1)}, \sigma_3][1+O(1/k)]\exp[ik(\sigma_3 x - \varepsilon y)]. \tag{3.4}$$

The notation $\pm$ means + when $\varepsilon^2 = +1$ ($\varepsilon = +1$) and − when $\varepsilon^2 = -1$ ($\varepsilon = -1$).

Therefore the quantity $\psi_x \pm \varepsilon\sigma_3\psi_y$ is a solution of (2.5a) and has an asymptotic behaviour proportional (independently of k) to ψ. Then if, as assumed, we are in the case when the solution of (2.6) is unique, we have

$$T_1\psi = 0 \qquad T_1 = \partial_x \pm \varepsilon\sigma_3\partial_y + Q \tag{3.5}$$

$$Q = -i[\phi^{(1)}(x, y), \sigma_3]. \tag{3.6}$$

The spectral problem (3.5) is the linear 2×2 first-order two-dimensional generalisation of the Zakharov-Shabat spectral problem and has been studied in [4]. For polynomial dispersion relations it allows us to solve the Davey-Stewartson system.

As regards the auxiliary spectral operator T_2, we select those dispersion relations Ω for which W vanishes on the boundary $\partial\mathscr{D}$ ($|k|\to\infty$). From (2.8) and (2.9),

$$W=\psi_t\psi^{-1}-\psi\Omega\psi^{-1}. \tag{3.7}$$

Choosing Ω diagonal as in one dimension we obtain

$$W\sim -\Omega \qquad \text{as } |k|\to\infty \tag{3.8}$$

and W will vanish with Ω. In summary, by selecting

$$\Omega=\omega(k,t)\sigma_3 \qquad \omega\sim O(1/k) \qquad \text{as } |k|\to\infty \tag{3.9}$$

we avoid the appearance of polynomial terms in the solution W of (2.10). But we have to stress that this is only made to focus on 'completely' singular dispersion relations for convenience and simplicity. It is actually possible to generalise the procedure to the case when Ω has a polynomial part. This is done in appendix 1.

Now, with the structure (3.5) of T_1, the weak compatibility condition (2.12) becomes

$$(\partial_x\pm\varepsilon\sigma_3\partial_y)W+Q_t+[Q,W]\pm\varepsilon[\sigma_3,W]G=0 \tag{3.10}$$

in which the quantity

$$G\doteq\psi_y\psi^{-1} \tag{3.11}$$

is easily shown to obey

$$(\partial_x\mp\varepsilon\sigma_3\partial_y)G+Q_y+[Q,G]\pm\varepsilon[\sigma_3,G]G=0 \tag{3.12}$$

with the behaviour

$$G\sim -\mathrm{i}\varepsilon k+O(1/k) \qquad \text{as } |k|\to\infty. \tag{3.13}$$

It is convenient to introduce the distributions

$$\Phi=\frac{\partial W}{\partial\bar{k}} \qquad \Gamma=\frac{\partial G}{\partial\bar{k}} \tag{3.14}$$

and the operator I_k inverse of $\partial\bar{k}$ given by

$$I_kA(k)\doteq\frac{1}{2\mathrm{i}\pi}\int\int\frac{\mathrm{d}l\wedge\mathrm{d}\bar{l}}{l-k}A(l). \tag{3.15}$$

We also define for future use the average

$$\langle A\rangle\doteq\frac{1}{2\mathrm{i}\pi}\int\int\mathrm{d}l\wedge\mathrm{d}\bar{l}\,A(l). \tag{3.16}$$

Using (3.13) and (3.8), the inverse of (3.14) is

$$W=I_k\Phi \qquad G=-\mathrm{i}\varepsilon k+I_k\Gamma. \tag{3.17}$$

With these notations in hand we may exploit (3.10) and (3.12) by first looking at the large-$|k|$ asymptotics. It gives the set

$$Q_t+\mathrm{i}[\sigma_3,\langle\Phi\rangle]=0 \tag{3.18}$$

$$Q_y+\mathrm{i}[\sigma_3,\langle\Gamma\rangle]=0. \tag{3.19}$$

Then we differentiate (3.10) and (3.12) with respect to $\bar{k}$ to obtain

$$(\partial_x \pm \varepsilon\sigma_3\partial_y)\Phi + [Q - ik\sigma_3, \Phi] = \mp\varepsilon[\sigma_3, \Phi]I_k\Gamma \mp \varepsilon[\sigma_3, I_k\Phi]\Gamma \tag{3.20}$$

$$(\partial_x \pm \varepsilon\sigma_3\partial_y)\Gamma + [Q - ik\sigma_3, \Gamma] = \mp\varepsilon[\sigma_3, \Gamma]I_k\Gamma \mp \varepsilon[\sigma_3, I_k\Gamma]\Gamma. \tag{3.21}$$

The above set of four coupled equations for the quantities Φ, Γ and Q constitutes the general form of the solvable non-linear evolution associated with the 2×2 linear first-order system in the plane for singular dispersion relation. The relevant equation is the evolution (3.18) giving the time dependence of the matrix field Q. This evolution is coupled to the spectral problem (3.5) for ψ through the matrix distribution ϕ defined by (3.14) and (3.7) which, as will be seen later, plays the role of the 'squared eigenfunction'. Instead of considering (3.18) as being coupled to the spectral problem (3.5) itself, we may consider it as being coupled to the system (3.20) and (3.21). Then the equation (3.19) is automatically verified. We shall see in the next section how equations (3.20) and (3.21) which are written for distributions can actually be written for functions.

This set of equations has to be completed with some boundary/initial values. Until now we have considered boundary values in the complex k plane only but it is more convenient (for physical reasons) to associate to the system (3.18)-(3.21) a set of boundary conditions in the space (xy) plane.

The field $Q(x, y, t)$ is constrained to obey

$$Q \to 0 \qquad \text{as } x^2 + y^2 \to \infty \tag{3.22}$$

and it is shown in the following sections that

$$\Phi \to -\frac{\partial\Omega}{\partial\bar{k}} \qquad W \to -\Omega \qquad \text{as } x \to -\infty \tag{3.23}$$

$$\Gamma \to 0 \qquad G \to i\varepsilon k \qquad \text{as } x \to -\infty \tag{3.24}$$

at least in the sense of distributions.

To prove these behaviours it is necessary to study separately the two cases $\varepsilon^2 = \pm 1$ and to use explicitly the corresponding particular form of the $\bar{\partial}$ equation (2.5*a*) where R will appear to be an off-diagonal matrix. Both $\bar{\partial}$ problems and their solutions are recalled for completeness in appendix 2. The solutions have been originally given in [4].

A better insight into what is contained in equations (3.18)-(3.21) is provided by reducing them to one spatial dimension. In that case

$$Q_y = 0 \qquad \psi(k, x, y) = \psi_0(k, x)\, e^{-i\varepsilon k y} \qquad G = -i\varepsilon k \qquad \Gamma = 0. \tag{3.25}$$

Then (3.21) and (3.19) disappear and the right-hand side of (3.20) vanishes. Still in that case (see (2.16) and (3.9))

$$\Phi = -\psi\sigma_3\psi^{-1}\frac{\partial\omega}{\partial\bar{k}} = -\psi_0\sigma_3\psi_0^{-1}\frac{\partial\omega}{\partial\bar{k}} \tag{3.26}$$

and therefore Φ_y vanishes. Finally the system (3.18)-(3.21) reduces to the couple

$$Q_t + i[\sigma_3, \langle\Phi\rangle] = 0 \tag{3.27}$$

$$\Phi_x + [Q - ik\sigma_3, \Phi] = 0 \tag{3.28}$$

identical to that of [7] (or [10] in a reduced case) where we can see that (3.28) is the spectral problem for the squared eigenfunction $\psi\sigma_3\psi^{-1}$ (factorise the distribution $\partial\omega/\partial\bar{k}$ in (3.28)).

Therefore from one to two dimensions we have schematically described the following stages.

(i) Define two matrices Φ and Γ (distributions) analogue of the squared eigenfunctions (multiplied by $\partial\omega/\partial\bar{k}$).

(ii) Complete the t evolution of Q related with $\langle\Phi\rangle$ (same as in one dimension) with the 'y evolution' of Q related with $\langle\Gamma\rangle$.

(iii) Replace the polynomial spectral problem (3.28) for the squared eigenfunctions with two weakly coupled 'singular-integral-spectral' problems (3.20) and (3.21) for the new squared eigenstates Φ and Γ.

We may now sketch the method of solution. Given the data of $Q(x, y, 0)$ obeying (3.22) and the diagonal matrix distribution $\partial\Omega/\partial\bar{k}$, the problem consists in solving (3.18)-(3.21) with the boundary conditions (3.22)-(3.24).

This is performed by solving first the direct spectral problem for (3.5) to obtain from $Q(x, y, 0)$ the eigenstate $\psi(k, x, y, 0)$ and the spectral transform $R(k, l, 0)$ (see appendix 2).

Then $R(k, l, t)$ is obtained from $R(k, l, 0)$ by solving the linear equation (2.7) where Ω is given in terms of $\partial\Omega/\partial\bar{k}$ by

$$\Omega(k, t)=\frac{1}{2\mathrm{i}\pi}\int\int\frac{\mathrm{d}l\wedge\mathrm{d}\bar{l}}{l-k}\frac{\partial\Omega(l, t)}{\partial\bar{l}}. \tag{3.29}$$

$Q(x, y, t)$ and $\psi(k, x, y, t)$ are obtained by solving an appropriate integral equation for ψ (see appendix 2).

With $\psi(k, x, y, t)$, W and G are readily computed by means of (3.7) and (3.11) and hence Φ and Γ by means of (3.14)

We note finally that the main information, that is the function $\Omega(k, t)$, is contained in the asymptotic behaviour (3.23) of Φ.

In concluding this section we remark that the construction of the solvable evolution for the singular dispersion relation for different spectral problems now becomes an exercise. The method that we have developed readily applies as soon as one has the $\bar{\partial}$ problem and the corresponding spectral problem.

4. Hyperbolic case ($\varepsilon = 1$)

We examine here in more detail the evolution equations and the evolution of the spectral transform when $\varepsilon = 1$, that is when the spectral problem is

$$T_1\psi = 0 \qquad T_1 = \partial_x + \sigma_3\partial_y + Q. \tag{4.1}$$

In [4] the spectral problem for (4.1) is written under the form of a Riemann-Hilbert problem. We shall see in appendix 2 that it can be written also as a $\bar{\partial}$ problem (2.5) with the following particular form of $R(k, l)$ when Q obeys (2.22) (as in [4] we suppose that the integral equation has no homogeneous solution):

$$R(k, l)=\frac{\mathrm{i}}{2}\begin{pmatrix}0 & r_1(k, l)\delta(l_I+0)\delta(k_I+0)\\ r_2(k, l)\delta(l_I-0)\delta(k_I-0) & 0\end{pmatrix} \tag{4.2}$$

$$k = k_R + \mathrm{i}k_I \qquad l = l_R + \mathrm{i}l_I.$$

In the following we shall denote matrix $\begin{pmatrix}0 & r_1\\ r_2 & 0\end{pmatrix}$ by R_1.

The first thing to do now is to prove that the asymptotic behaviours (3.23) and (3.24) hold. When ψ is fixed by its behaviour as $|k| \to \infty$ as in (3.3), it is possible to prove [4] that we have also

$$\psi(k, x, y, t) \to \exp[\mathrm{i}k\sigma_3 x - \mathrm{i}ky] \qquad \text{as } x \to -\infty. \tag{4.3}$$

Therefore G defined in (3.11) obeys

$$G(k, x, y, t) \to -\mathrm{i}k \qquad \text{as } x \to -\infty \tag{4.4}$$

and W defined in (3.7)

$$W(k, x, y, t) \to -\Omega(k, t) \qquad \text{as } x \to -\infty. \tag{4.5}$$

The distributions $\Phi = \partial W/\partial \bar{k}$ given in (2.10) and $\Gamma = \partial G/\partial \bar{k}$ become after making use of (4.2) (we forget the x, y, t dependence)

$$\Phi(k) = \delta(k_I) \int_{-\infty}^{+\infty} \mathrm{d}l\, [W(l) - W(k)]\psi(l) R_1(k, l)\psi^{-1}(k) - \psi(k)\frac{\partial \Omega}{\partial \bar{k}}\psi^{-1}(k) \tag{4.6}$$

$$\Gamma(k) = \delta(k_I)\Gamma_0(k) \tag{4.7}$$

$$\Gamma_0(k) = \int_{-\infty}^{+\infty} \mathrm{d}l\, \psi_y(l) R_1(k, l)\psi^{-1}(k) - \psi_y(k)\psi^{-1}(k) \int_{-\infty}^{+\infty} \mathrm{d}l\, \psi(l) R_1(k, l)\psi^{-1}(k). \tag{4.8}$$

These formulae are written in shorthand and we have to remember that, for instance, the first column of $W(l)\psi(l)$ is actually taken at $l + \mathrm{i}0$ and the second at $l - \mathrm{i}0$. In the same way (4.6) and (4.8) have to be read for $k = k_r \pm \mathrm{i}0$.

By taking now the limit as $x \to -\infty$ of (4.6) and (4.8) and using (4.3), the off-diagonal character of R_1 and the diagonality of Ω imply that the behaviours (3.23) and (3.24) do hold in the sense of functions.

As mentioned in the preceding section, equations (3.20) and (3.21) are written for the distributions Φ and Γ and therefore are not convenient to describe a physical situation. But in (4.7) Γ is factorised which allows us to write (3.21) for the *function* $\Gamma_0(k)$. Equation (3.20) with Φ given in (4.6) will be factorised in the same way if we choose

$$\frac{\partial \Omega}{\partial \bar{k}} = g(k, t)\sigma_3 \delta(k_I) \tag{4.9}$$

or else

$$\Omega^{\pm}(k, t) = \mp \mathrm{i}g(k, t)\sigma_3 - \frac{1}{\pi}\sigma_3 \fint_{-\infty}^{+\infty} \frac{\mathrm{d}l}{l - k_R} g(l, t) \qquad k = k_R \pm \mathrm{i}0 \tag{4.10}$$

where $\fint_{-\infty}^{+\infty}$ denotes the Cauchy principal value integral. Then we have

$$\Phi(k) = \delta(k_I)\Phi_0(k) \tag{4.11}$$

$$\Phi_0(k) = -g\psi(k)\sigma_3\psi^{-1}(k) + \int_{-\infty}^{+\infty} \mathrm{d}l\, [W(l) - W(k)]\psi(l) R_1(k, l)\psi^{-1}(k) \tag{4.12}$$

and the set of solvable evolutions (3.18)-(3.21) becomes

$$Q_t + \mathrm{i}[\sigma_3, \langle \Phi_0 \rangle] = 0 \tag{4.13}$$

$$Q_y + \mathrm{i}[\sigma_3, \langle \Gamma_0 \rangle] = 0 \tag{4.14}$$

$$(\partial_x + \sigma_3 \partial_y)\Phi_0 + [Q - \mathrm{i}k\sigma_3, \Phi_0] = -[\sigma_3, \Phi_0] I\Gamma_0 - [\sigma_3, I\Phi_0]\Gamma_0 \tag{4.15}$$

$$(\partial_x + \sigma_3 \partial_y)\Gamma_0 + [Q - \mathrm{i}k\sigma_3, \Gamma_0] = -[\sigma_3, \Gamma_0] I\Gamma_0 - [\sigma_3, I\Gamma_0]\Gamma_0 \tag{4.16}$$

where now we define

$$\langle\Phi_0\rangle = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \mathrm{d}l\, \Phi_0(l) \qquad I\Phi_0 = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{d}l}{l-k}\,\Phi_0(l). \tag{4.17}$$

With these equations the following behaviours are given:

$$Q \to 0 \qquad \text{as } x^2+y^2 \to \infty \tag{4.18}$$

$$\Gamma_0 \to 0 \qquad \text{as } x \to -\infty \tag{4.19}$$

$$\Phi_0 \to g(k,t)\sigma_3 \qquad \text{as } x \to -\infty. \tag{4.20}$$

The set of equations (4.13)-(4.20) is solvable with the method given in § 3. The corresponding evolution of the spectral transform is obtained from (2.7) and (4.2) and is, for $k, l \in \mathbb{R}$,

$$R_1(k,l,t) = R_1(k,l,0)\sigma_3 \int_0^t \mathrm{d}t \begin{pmatrix} \omega^+(k,t)+\omega^+(l,t) & 0 \\ 0 & \omega^-(k,t)+\omega^-(l,t) \end{pmatrix} \tag{4.21}$$

where $\omega^{\pm}$ is

$$\omega^{\pm}(k,t) = \mp \mathrm{i}g(k,t) - \frac{1}{\pi}\not\int_{-\infty}^{+\infty} \frac{\mathrm{d}l}{l-k}\, g(l,t). \tag{4.22}$$

Depending on $g(k)$ (asymptotic behaviour of Φ in (4.20)) the time evolution of R_1 may present exponential growth or damping. In 1+1 dimensions the exponential damping causes the self-induced transparency phenomenon [10] and the exponential growth allows us to consider theoretical models for laser amplifiers [15, 16]. These phenomena then occur also in 2+1 dimensions.

5. Elliptic case ($\varepsilon = -\mathrm{i}$)

We now consider briefly the case when

$$T_1\psi = 0 \qquad T_1 = \partial_x + \mathrm{i}\sigma_3\partial_y + Q. \tag{5.1}$$

In the $\bar\partial$ problem (2.5), the spectral transform R has the following form (see [4] and appendix 2):

$$R(k,l) = \delta(l-\bar k)R_2(k,\bar k) \tag{5.2}$$

where R_2 is an off-diagonal matrix.

The behaviours (4.3) and (4.4) become respectively

$$\psi \to \exp(\mathrm{i}k\sigma_3 x - ky)$$

and

$$G \to -k \qquad \text{as } x \to -\infty$$

and the behaviour (4.5) still holds. Now we have

$$\Phi(k) = [W(\bar k) - W(k)]\psi(\bar k)R_2(k,\bar k)\psi^{-1}(k) - \psi(k)(\partial\Omega/\partial\bar k)\psi^{-1}(k) \tag{5.3}$$

$$\Gamma(k) = \psi_y(\bar k)R_2(k,\bar k)\psi^{-1}(k) - \psi_y(k)\psi^{-1}(k)\psi(\bar k)R_2(k,\bar k)\psi^{-1}(k) \tag{5.4}$$

and hence ($k = k_R + \mathrm{i}k_I$)

$$\Phi(k) \to -[\Omega(\bar k) - \Omega(k)]R_2(k,\bar k)\,\mathrm{e}^{-2\mathrm{i}\sigma_3 k_R x}\,\mathrm{e}^{2\mathrm{i}k_I y} - \partial\Omega/\partial\bar k \tag{5.5}$$

$$\Gamma(k) \to (k-\bar k)R_2(k,\bar k)\,\mathrm{e}^{-2\mathrm{i}\sigma_3 k_R x}\,\mathrm{e}^{2\mathrm{i}k_I y}. \tag{5.6}$$

One may check that these behaviours are indeed compatible with the evolutions (3.20) and (3.21) when $x \to -\infty$, as soon as R_2 anticommutes with σ_3 and Ω commutes with σ_3.

The above asymptotic behaviours verify (3.23) and (3.24) only in the sense of distributions. However, they are bounded and their main property is that they depend on the spectral transform R_2 which itself can be expressed in terms of ψ and Q. Thus we have here a kind of new non-linear asymptotic behaviour. Note that these behaviours do not affect the evolutions (3.18) and (3.19) where Φ and Γ are integrated on $\mathbb{C}$ through the definition (3.16) of the average.

By choosing

$$\partial\Omega/\partial\bar{k} = h(k, \bar{k}, t)\sigma_3 \tag{5.7}$$

equations (3.20) and (3.21) are directly written in terms of functions.

The set (3.18)-(3.21) with the boundary conditions (5.5) and (5.6) is integrable with the method of § 3. The corresponding evolution of the spectral transform gives

$$R_2(k, \bar{k}, t) = \exp\left(-\int_0^t \mathrm{d}t\, I_{\bar{k}} h\sigma_3\right) R_2(k, \bar{k}, 0) \exp\left(\int_0^t \mathrm{d}t\, I_k h\sigma_3\right). \tag{5.8}$$

(The operator I_k is defined in (3.15).)

6. Bäcklund transformations and soliton solutions

The general method to establish and solve (for soliton solutions) the Bäcklund transformations in 2+1 dimensions is given in detail in [17]. In short the Bäcklund gauge B (operator) which links the two potentials Q and Q' and their corresponding eigenstates ψ and ψ' via

$$\psi' = B\psi N \tag{6.1}$$

is determined up to some constants of integration by the operator equation

$$T_1' B - B T_1 = 0. \tag{6.2}$$

In (6.1) N is a constant (k-dependent) matrix used to normalise ψ' as in (2.5b). The requirement that Q and Q' solve the same evolution is equivalent to the equation

$$T_2' B - B T_2 = 0 \tag{6.3}$$

and allows us to determine the time dependence of the constants of integration.

The simplest non-trivial Bäcklund transformations are obtained by solving (6.2) and (6.3) for the choice

$$B = \alpha\partial_y + B_1 \tag{6.4}$$

where α is a constant diagonal matrix.

Inserting (6.4) into (6.2) with T_1 given in (3.5) allows us to prove that

$$B_1 = -\tfrac{1}{2}\varepsilon\sigma_3(Q'\alpha - \alpha Q) - \tfrac{1}{2}\varepsilon\sigma_3\alpha J_\varepsilon(Q'^2 - Q^2) + \beta \tag{6.5}$$

where β is a constant diagonal matrix, and gives the Bäcklund transformation

$$-\tfrac{1}{2}\varepsilon\sigma_3(\partial_x \pm \varepsilon\sigma_3\partial_y)(Q'\alpha - \alpha Q) + Q'\beta - \beta Q - \alpha Q_y$$
$$+\tfrac{1}{2}\varepsilon\sigma_3\{\alpha[J_\varepsilon(Q'^2 - Q^2)]Q + Q'\alpha J_\varepsilon(Q'^2 - Q^2)\} = 0. \tag{6.6}$$

In the above equations the integral operator J_ε is defined by its action on a generic diagonal matrix by

$$J_\varepsilon \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \int_{-\infty}^{x} \mathrm{d}x' \begin{pmatrix} a[x', y \pm \varepsilon(x'-x)] & 0 \\ 0 & b[x', y \pm \varepsilon(x'-x)] \end{pmatrix} \tag{6.7}$$

such that

$$(\partial_x \pm \varepsilon\sigma_3\partial_y) J_\varepsilon \equiv \mathbb{1}. \tag{6.8}$$

The equation (6.6) is called the 'x part' of the Bäcklund transformation and does not depend on the operator T_2. Choosing T_2 as in (2.8), (6.3) then gives the so-called 't part'

$$\alpha_t G + B_{1,t} = (W'\alpha - \alpha W)G + W'B_1 - B_1 W - \alpha W_y. \tag{6.9}$$

We recall that as $x \to -\infty$ we have

$$B \to \alpha\partial_y + \beta \qquad G \to -\mathrm{i}\varepsilon k \qquad W \to -\omega\sigma_3 \qquad W' \to -\omega\sigma_3. \tag{6.10}$$

(In T_2 and T_2' we choose the *same* dispersion relation $\omega\sigma_3$ to ensure that Q and Q' obey the *same* evolution.) Taking the limit of (6.9) as $x \to -\infty$, we obtain

$$\alpha_t = 0 \qquad \beta_t = 0. \tag{6.11}$$

For future use we then express (6.9) in the following form:

$$W' = [B_{1,t} + B_1 W + \alpha(WG + W_y)][B_1 + \alpha G]^{-1}. \tag{6.12}$$

The two elementary Bäcklund transformations are obtained for the two choices

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{'type I'} \tag{6.13}$$

$$\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \qquad \text{'type II'} \tag{6.14}$$

where λ and μ are arbitrary constants.

A successive application of these two elementary transformations allows us to build recursively a whole lattice of solutions [17]. We are going now to derive the explicit formulae when the starting potential Q vanishes. To that end we have to consider separately the two cases $\varepsilon^2 = \pm 1$.

6.1. Hyperbolic case ($\varepsilon = 1$)

In the case $Q = 0$ the solutions of (6.6) for the two elementary structures (6.13) and (6.14) are respectively (we drop the primes whenever it is not ambiguous)

$$Q^{(\mathrm{I})} = \begin{pmatrix} 0 & 0 \\ \rho(x+y, t)\, \mathrm{e}^{-(x-y)\lambda} & 0 \end{pmatrix} \qquad \mathrm{Re}\,\lambda < 0 \tag{6.15}$$

$$Q^{(\mathrm{II})} = \begin{pmatrix} 0 & \eta(x-y, t)\, \mathrm{e}^{(x+y)\mu} \\ 0 & 0 \end{pmatrix} \qquad \mathrm{Re}\,\mu > 0 \tag{6.16}$$

where ρ and η are arbitrary functions to be determined and where the signs of the real parts of λ and μ are chosen such that Q vanishes as $x \to -\infty$.

The corresponding gauge operators are then obtained from (6.4) and (6.5)

$$B^{(\mathrm{I})}=\begin{pmatrix}\lambda+\partial_y & 0\\ \frac{1}{2}\rho\, \mathrm{e}^{-(x-y)\lambda} & 1\end{pmatrix} \tag{6.17}$$

$$B^{(\mathrm{II})}=\begin{pmatrix}1 & -\frac{1}{2}\eta\, \mathrm{e}^{(x+y)\mu}\\ 0 & \mu+\partial_y\end{pmatrix}. \tag{6.18}$$

The usual standard method to compute ρ and η is to use the t part (6.9) of the transformation. Here, however, this method is useless because W' is not explicitly given in terms of Q'. To perform the computations we shall use a different approach which proceeds through the following stages.

(i) Compute the eigenfunction ψ' for the potentials $Q^{(\mathrm{I})}$ and $Q^{(\mathrm{II})}$.

(ii) Compute the spectral transforms by evaluating $\partial\psi/\partial\bar{k}$. This will give both $R(k, l)$ and the structure of the xy dependence of ρ and η.

(iii) Make these spectral transforms evolve in time according to (2.7) which furnishes the time dependence of ρ and η and achieves the derivation of the elementary Bäcklund transformations.

We note first the following relations:

$$Q=0\Rightarrow\{\psi=\mathrm{e}^{\mathrm{i}k(\sigma_3 x-y)},\ G=-\mathrm{i}k,\ W=-\omega\sigma_3\}. \tag{6.19}$$

Then the Jost solutions $\psi^{(\mathrm{I})}$ and $\psi^{(\mathrm{II})}$ are obtained from the definition (6.1) of the gauge operator B. With appropriate choice of N we get

$$\psi^{(\mathrm{I})}=\mathrm{e}^{\mathrm{i}k(\sigma_3 x-y)}+\frac{\mathrm{i}}{2(k+\mathrm{i}\lambda)}\,\mathrm{e}^{\mathrm{i}(k+\mathrm{i}\lambda)(x-y)}\rho\sigma_- \tag{6.20}$$

$$\psi^{(\mathrm{II})}=\mathrm{e}^{\mathrm{i}k(\sigma_3 x-y)}-\frac{\mathrm{i}}{2(k+\mathrm{i}\mu)}\,\mathrm{e}^{-\mathrm{i}(k+\mathrm{i}\mu)(x+y)}\eta\sigma_+ \tag{6.21}$$

$$\sigma_+=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\qquad \sigma_-=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

Hence

$$\frac{\partial}{\partial\bar{k}}\psi^{(\mathrm{I})}=\pi\delta(k+\mathrm{i}\lambda)\rho\sigma_- \tag{6.22}$$

$$\frac{\partial}{\partial\bar{k}}\psi^{(\mathrm{II})}=-\pi\delta(k+\mathrm{i}\mu)\eta\sigma_+. \tag{6.23}$$

We use now explicitly (2.5), in which we replace ψ by $\psi^{(\mathrm{I})}$ and $\psi^{(\mathrm{II})}$, with the following definitions:

$$\rho(x+y, t)\doteq\int\int \mathrm{d}l\wedge\mathrm{d}\bar{l}\,\mathrm{e}^{-\mathrm{i}l(x+y)}\tilde{\rho}(l, t) \tag{6.24}$$

$$\eta(x-y, t)\doteq\int\int \mathrm{d}l\wedge\mathrm{d}\bar{l}\,\mathrm{e}^{\mathrm{i}l(x-y)}\tilde{\eta}(l, t) \tag{6.25}$$

to obtain the following spectral transforms of $Q^{(\mathrm{I})}$ and $Q^{(\mathrm{II})}$:

$$R^{(\mathrm{I})}(k, l)=\pi\delta(k+\mathrm{i}\lambda)\tilde{\rho}(l, t)\sigma_- \tag{6.26}$$

$$R^{(\mathrm{II})}(k, l)=-\pi\delta(k+\mathrm{i}\mu)\tilde{\eta}(l, t)\sigma_+. \tag{6.27}$$

Note that the structure (4.2) of R does not hold here because the potentials $Q^{(\mathrm{I})}$ and $Q^{(\mathrm{II})}$ do not verify (3.22).

The time evolution (2.7) for $\Omega = \omega\sigma_3$ readily gives

$$\tilde{\rho}(l, t) = \tilde{\rho}(l, 0) \exp\left(\int_0^t \mathrm{d}t \, \{\omega(l, t) + \omega(-\mathrm{i}\lambda, t)\} \right) \tag{6.28}$$

$$\tilde{\eta}(l, t) = \tilde{\eta}(l, 0) \exp\left(-\int_0^t \mathrm{d}t \, \{\omega(l, t) + \omega(-\mathrm{i}\mu, t)\} \right) \tag{6.29}$$

which achieves the construction of $Q^{(\mathrm{I})}$ and $Q^{(\mathrm{II})}$ together with their eigenfunctions $\psi^{(\mathrm{I})}$ and $\psi^{(\mathrm{II})}$. Note that the spectrum of $Q^{(\mathrm{I})}$ ($Q^{(\mathrm{II})}$) consists in one discrete eigenvalue located in the upper half-plane: $k = -\mathrm{i}\lambda$ (the lower half-plane: $k = -\mathrm{i}\mu$) as seen on (6.26) and (6.27), remembering that $\mathrm{Re}\,\lambda < 0$ and $\mathrm{Re}\,\mu > 0$.

To complete the derivation of the solution of the non-linear evolution (3.18)-(3.21) we need G' and W'. $G' = \psi'_y \psi'^{-1}$ is obviously computed from $\psi^{(\mathrm{I})}$ and $\psi^{(\mathrm{II})}$ and W' is given by (6.12) in terms of B_1 (obtained in (6.17) and (6.18)), α and the couple (G, W) given in (6.19). We get

$$W^{(\mathrm{I})} = -\omega\sigma_3 + \frac{\mathrm{i}}{2(k+\mathrm{i}\lambda)} [Q_t^{(\mathrm{I})} - 2\omega Q^{(\mathrm{I})}] \tag{6.30}$$

$$W^{(\mathrm{II})} = -\omega\sigma_3 - \frac{\mathrm{i}}{2(k+\mathrm{i}\mu)} [Q_t^{(\mathrm{II})} + 2\omega Q^{(\mathrm{II})}]. \tag{6.31}$$

It is then possible to check that $W^{(\mathrm{I})}$ and $W^{(\mathrm{II})}$ are indeed solutions of (2.10) in which $R^{(\mathrm{I})}$ and $R^{(\mathrm{II})}$ are given in (6.26) and (6.27) and $\psi^{(\mathrm{I})}$ and $\psi^{(\mathrm{II})}$ in (6.20) and (6.21).

6.2. Elliptic case ($\varepsilon = -i$)

The method being exactly the same as for the hyperbolic case we only list below the relevant formulae (ρ and η and two other arbitrary functions)

$$Q^{(\mathrm{I})} = \rho(\mathrm{i}x + y, t) \, \mathrm{e}^{\lambda(-\mathrm{i}x+y)}\sigma_- \qquad \mathrm{Im}\,\lambda > 0 \tag{6.32}$$

$$Q^{(\mathrm{II})} = \eta(\mathrm{i}x - y, t) \, \mathrm{e}^{\mu(\mathrm{i}x+y)}\sigma_+ \qquad \mathrm{Im}\,\mu < 0 \tag{6.33}$$

$$\psi^{(\mathrm{I})} = \mathrm{e}^{\mathrm{i}k\sigma_3 x - ky} + \frac{\mathrm{i}}{2(k-\lambda)} \mathrm{e}^{(k-\lambda)(\mathrm{i}x-y)}\rho\sigma_- \tag{6.34}$$

$$\psi^{(\mathrm{II})} = \mathrm{e}^{\mathrm{i}k\sigma_3 x - ky} - \frac{\mathrm{i}}{2(k-\mu)} \mathrm{e}^{-(k-\mu)(\mathrm{i}x+y)}\eta\sigma_+ \tag{6.35}$$

$$\rho = \rho_0(t) \, \mathrm{e}^{-\bar{\lambda}(\mathrm{i}x+y)} \tag{6.36}$$

$$\eta = \eta_0(t) \, \mathrm{e}^{\bar{\mu}(\mathrm{i}x-y)} \tag{6.37}$$

$$R^{(\mathrm{I})} = \pi\delta(k-\lambda)\rho_0(t)\sigma_- \tag{6.38}$$

$$R^{(\mathrm{II})} = -\pi\delta(k-\mu)\eta_0(t)\sigma_+ \tag{6.39}$$

$$\rho_0(t) = \rho_0(0) \exp\left(\int_0^t \omega(\lambda, t) + \omega(\bar{\lambda}, t) \right) \tag{6.40}$$

$$\eta_0(t) = \eta_0(0) \exp\left(-\int_0^t \omega(\mu, t) + \omega(\bar{\mu}, t) \right) \tag{6.41}$$

$$W^{(\mathrm{I})} = -\omega\sigma_3 + \frac{\mathrm{i}}{2(k-\lambda)} \left(Q_t^{(\mathrm{I})} - 2\omega Q^{(\mathrm{I})} \right) \tag{6.42}$$

$$W^{(\mathrm{II})} = -\omega\sigma_3 - \frac{\mathrm{i}}{2(k-\mu)} \left(Q_t^{(\mathrm{II})} + 2\omega Q^{(\mathrm{II})} \right). \tag{6.43}$$

6.3. Non-linear superposition

It is well known that the non-linear superposition principle (or Bianchi theorem) allows us to build a solution from three known ones through simple algebraic steps. Moreover the formula is derived in [17] for evolutions in 2+1 dimensions related to the spectral problem (3.5) and makes no reference to the time evolution. We shall therefore only sketch the results.

The gauge operator B which allows us to go from the vanishing solution to the one-soliton solution is given by

$$B \doteq B_{31}^{(\mu)} B_{10}^{(\lambda)} = B_{32}^{(\lambda)} B_{20}^{(\mu)}. \tag{6.44}$$

In the above formula $B_{ij}^{(\lambda)}$ ($B_{ij}^{(\mu)}$) represents the elementary gauge operator of type I (type II) which allows us to obtain the potential Q_i from the potential Q_j [17].

With the notation $Q_1 = Q^{(1)}$, $Q_2 = Q^{(11)}$ and

$$Q_3 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \qquad Q_1 = \begin{pmatrix} 0 & 0 \\ r_1 & 0 \end{pmatrix} \qquad Q_2 = \begin{pmatrix} 0 & q_2 \\ 0 & 0 \end{pmatrix} \tag{6.45}$$

we obtain from (6.44) the following soliton solution:

$$q = \frac{q_{2,y} + (\lambda - \mu) q_2}{1 + \frac{1}{4}\varepsilon^2 r_1 q_2} \qquad r = \frac{r_{1,y} - (\lambda - \mu) r_1}{1 + \frac{1}{4}\varepsilon^2 r_1 q_2}. \tag{6.46}$$

The corresponding eigenfunction is obtained by using (6.1) for B given by (6.44) as

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_y + \begin{pmatrix} \lambda - \frac{1}{4}\varepsilon^2 r_1 q & -\frac{1}{2}\varepsilon q \\ \frac{1}{2}\varepsilon r & \mu - \frac{1}{4}\varepsilon^2 q_2 r \end{pmatrix} \tag{6.47}$$

and for the following appropriate choice of N:

$$N = \begin{pmatrix} \dfrac{1}{\lambda - \mathrm{i}\varepsilon k} & 0 \\ 0 & \dfrac{1}{\mu - \mathrm{i}\varepsilon k} \end{pmatrix}. \tag{6.48}$$

This eigenfunction finally becomes

$$\psi = \begin{pmatrix} 1 - \frac{1}{4}\varepsilon^2 \dfrac{r_1 q}{(\lambda - \mathrm{i}\varepsilon k)} & -\frac{1}{2}\varepsilon \dfrac{q}{(\mu - \mathrm{i}\varepsilon k)} \\ \dfrac{\varepsilon r}{2(\lambda - \mathrm{i}\varepsilon k)} & 1 - \frac{1}{4}\varepsilon^2 \dfrac{q_2 r}{(\mu - \mathrm{i}\varepsilon k)} \end{pmatrix} \mathrm{e}^{\mathrm{i}k(\sigma_3 x - \varepsilon y)} \tag{6.49}$$

and the corresponding spectral transforms are

$$R(\varepsilon = 1) = \mathrm{i}\pi \begin{pmatrix} 0 & \delta(k + \mathrm{i}\mu)\tilde{\eta}(l, t)(l + \mathrm{i}\lambda) \\ -\delta(k + \mathrm{i}\lambda)\,\tilde{\rho}(l, t)(l + \mathrm{i}\mu) & 0 \end{pmatrix} \tag{6.50}$$

$$R(\varepsilon = -\mathrm{i}) = \pi \begin{pmatrix} 0 & -\delta(\mu - k)\eta_0(t)(\lambda - \bar{k}) \\ \delta(\lambda - k)\rho_0(t)(\mu - \bar{k}) & 0 \end{pmatrix}. \tag{6.51}$$

The one-soliton solution of the system (3.18)–(3.21) is completed with the computation of $G' = \psi_y \psi^{-1}$ and $\Gamma' = (\partial/\partial \bar{k}) G'$ (ψ given in (6.49)), then of W' through (4.12) where B_1 is obtained in (6.47), $W = -\omega\sigma_3$, $\alpha = \mathbb{1}$ and $G = -\mathrm{i}k(\varepsilon = -\mathrm{i})$ or $-k(\varepsilon = 1)$, and finally of $\Phi' = \partial W'/\partial \bar{k}$.

As shown in [17] one may then construct a whole lattice of solutions by superimposing arbitrary numbers of elementary transformations of the two types.

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Appendix 1. Generalisations of the evolutions

Instead of choosing a vanishing dispersion relation as in (3.9) we may set

$$\Omega = \omega\sigma_3 \qquad \omega = \omega_s + \omega_p \qquad \omega_s \sim O(1/k) \qquad |k| \to \infty \tag{A1.1}$$

where ω_p is some given polynomial in k.

Then T_2 is sought in the form

$$T_2 = \partial_t - W - \hat{V} \tag{A1.2}$$

where $W \simeq W(k, x, y, t)$ is a matrix and $\hat{V} = \hat{V}(\partial_y, x, y, t)$ is a polynomial differential operator in ∂_y.

The weak compatibility condition (2.4) gives

$$\{Q_t + [T_1, W] + [T_1, \hat{V}]\}\psi = 0 \tag{A1.3}$$

for

$$T_1 = \partial_x \pm \varepsilon\sigma_3\partial_y + Q \tag{A1.4}$$

and with the requirements analogous to (4.5):

$$W \to -\omega_s\sigma_3 \qquad \text{as } x \to -\infty \tag{A1.5}$$

$$\hat{V}\psi \sim -\omega_p\sigma_3 \exp[ik(\sigma_3 x - \varepsilon y)] \qquad \text{as } x \to -\infty. \tag{A1.6}$$

The matrix W is computed exactly as in the preceding sections (solution of (2.10)) and the operator $\hat{V}$ is computed as usual by asking that for

$$[T_1, \hat{V}]\psi = \sum_{j=0}^{n} C^{(j)}(x, y, t)\frac{\partial^{n-j}}{\partial_y^{n-j}}\psi \tag{A1.7}$$

we have, together with (A1.6),

$$C^{(j)} = 0 \qquad j = 0, \ldots, n-1. \tag{A1.8}$$

Equation (A1.8) allows us to compute recursively $\hat{V}$ in terms of Q.

The resulting evolution for Q is

$$Q_t + C^{(n)} + i[\sigma_3, \langle\Phi\rangle] = 0. \tag{A1.9}$$

This equation associated with (3.19)-(3.21) and the boundary conditions (3.23) and (3.24) (note that $\partial\Omega/\partial\bar{k} = \sigma_3\partial\omega_s/\partial\bar{k}$) constitutes the most general solvable evolution. In (A1.9) the quantity $Q_t + C^{(n)}$ simply stands for any equation of the (polynomial) hierarchy of non-linear evolution equations associated with T_1 (see [4, 18]).

Appendix 2. $\bar{\partial}$ problem and Riemann–Hilbert problem

We give hereafter the essential formulae obtained in [4] when the spectral operator T_1 is given in (3.5). The main point is to extract from [4] the formulae that we need, namely (4.2) and (5.2), in our notation.

In the hyperbolic case ($\varepsilon = 1$) it is shown in [4] that the function

$$\mu(k, x, y) = \psi(k, x, y) \exp[-ik(\sigma_3 x - y)] \tag{A2.1}$$

tending to $\mathbb{1}$ as $|k| \to \infty$ can be constructed by solving the following Riemann–Hilbert problem on the real axis:

$$\mu^+(k) - \mu^-(k) = \int_{-\infty}^{+\infty} dl\, \mu^-(l)\, e^{il(\sigma_3 x - y)} \mathcal{F}(l, k)\, e^{-ik(\sigma_3 x - y)} \tag{A2.2}$$

$$\mathcal{F}(l, k) = \begin{pmatrix} f_{11}(l, k) & f_{12}(l, k) \\ f_{21}(l, k) & 0 \end{pmatrix} \tag{A2.3}$$

$$f_{11}(l, k) = -\int_{-\infty}^{+\infty} dm\, f_{21}(m, k) f_{12}(l, m). \tag{A2.4}$$

The direct spectral problem, namely the construction of the spectral data, is solved [4] by using

$$\begin{pmatrix} 0 & f_{12}(l, k) \\ f_{21}(l, k) & 0 \end{pmatrix} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, Q(x, y) \begin{pmatrix} -\mu_{11}^+(k, x, y) & 0 \\ 0 & \mu_{22}^-(k, x, y) \end{pmatrix} \times \exp[i\sigma_3(l+k)x - i(l-k)y].$$

Our point is that the relation (A2.4) is precisely the condition that allows us to write (A2.2) as

$$\mu^+(k) - \mu^-(k) = \int_{-\infty}^{+\infty} dl\, (\mu_1^-(l), \mu_2^{(+)}(l))\, e^{il(\sigma_3 x - y)} \times \begin{pmatrix} 0 & f_{12}(l, k) \\ f_{21}(l, k) & 0 \end{pmatrix} e^{-ik(\sigma_3 x - y)} \tag{A2.5}$$

where μ_1 stands for the first column (and μ_2 the second) of the matrix μ. Rewriting finally (A2.5) for the original function ϕ and remembering the Sokhotski–Plemelj formula

$$\frac{\partial \psi}{\partial \bar{k}} = \frac{i}{2}[\psi(k_R + i0) - \psi(k_R - i0)] \tag{A2.6}$$

we obtain (remembering that $\mu^+ - \mu^-$ vanishes as $\alpha \to -\infty$ [4])

$$\frac{\partial \psi}{\partial \bar{k}} = \int\int dl \wedge d\bar{l}\, \psi(l) R(k, l) \tag{A2.7}$$

$$R(k, l) = -\tfrac{1}{4} \begin{pmatrix} 0 & f_{12}(l, k)\delta(l_I + 0)\, \delta(k_I + 0) \\ f_{21}(l, k)\delta(l_I - 0)\, \delta(k_I - 0) & 0 \end{pmatrix} \tag{A2.8}$$

and recover the structure given in (4.2).

Concerning the elliptic case, if we assume no homogeneous solution of the integral equation, the formula (4-21) of [4] written in terms of ψ gives directly

$$\frac{\partial \psi(k)}{\partial \bar{k}} = \psi(\bar{k}) R_2(k, \bar{k}) \tag{A2.9}$$

and therefore is obtained from (2.5) with the structure (5.2) of R.

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